

A BOUNDED NUMERICAL SOLUTION WITH A SMALL MESH SIZE INDICATES A SMOOTH SOLUTION OF THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS

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ABSTRACT. We prove that if the initial data is smooth enough and a finite element solution of the three-dimensional Navier–Stokes equations is bounded in a certain norm with a small mesh size, then the Navier–Stokes equations have a unique smooth solution.

1. Introduction

The Navier–Stokes equations have wide applications in many scientific and engineering fields, such as ocean currents, weather forecast, and air flow around a wing. Regardless of the wide range of their applications, whether the Navier–Stokes equations admit a unique solution is not known yet in three-dimensional domains for general smooth initial data. Existence of global weak solutions has been proved by Leray and Hopf a long time ago [14, 19], but it turns out to be very difficult to prove the smoothness of the weak solutions. Many recent efforts have been devoted to global well-posedness for small initial data [5, 16, 17, 18] and blowup examples of some Navier–Stokes type equations [6, 10, 15, 21, 27].

Driven by the various applications, many numerical methods have been proposed for solving the Navier–Stokes equations, such as the finite element methods [12, 13, 23], finite difference methods [7], spectral methods [11, 25], the Lagrange–Galerkin method [3, 22, 26], and the projection method for time discretization [8, 9, 24]. On one hand, conventionally, convergence of the numerical solutions were all proved by assuming that the Navier–Stokes equations have a unique smooth solution. Without assuming well-posedness of the Navier–Stokes equations, convergence of the numerical solutions cannot be proved a priori. On the other hand, numerical evidence of existence of blowup solutions for the Navier–Stokes equations is inadequate. In most practical situations, the numerical solution looks smooth and does not change much if the mesh is refined.

A natural question is, given a bounded numerical solution, what can we say about the smoothness of the true solution without assuming well-posedness of the Navier–Stokes equations? We answer this question in this paper: for any given smooth initial data, if the numerical solution is bounded in a certain norm for a small mesh size, then the Navier–Stokes equations have a unique smooth solution and, simultaneously, the numerical solution successfully approximates the solution of the Navier–Stokes equations. It is remarkable that we only need one numerical solution to draw the conclusion, instead

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of a sequence of numerical solutions. To illustrate our idea, we consider the Navier-Stokes equations

$$(1.1) \quad \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = 0,$$

$$(1.2) \quad \nabla \cdot \mathbf{u} = 0,$$

in a convex polyhedron $\Omega \subset \mathbb{R}^3$ with the Dirichlet boundary condition $\mathbf{u} = 0$ on $\partial\Omega$ and the initial condition $\mathbf{u}(x, 0) = \mathbf{u}_0(x)$ (where $\mu > 0$ is the viscosity constant), and we impose the condition $\int_{\Omega} p(x, t) dx = 0$ for uniqueness.

For simplicity, we focus on a simple linearized finite element method for the discretization of the Navier-Stokes equations. We hope that our methodology can be extended to other numerical methods as well as other nonlinear time-evolution equations for which global existence, uniqueness and regularity are unknown.

2. Notations and main results

Let the domain Ω be partitioned into quasi-uniform tetrahedra K_j , $j = 1, 2, \dots, J$, and denote by $h = \max_j \text{diam}(K_j)$ the spatial mesh size. For any positive integer r , let S_h^r be the space of globally continuous piecewise polynomials of degree r subject to the partition of the domain, and let \dot{S}_h^r be the subspace of S_h^r consisting of functions which are zero on the boundary $\partial\Omega$. For any nonnegative integer k , we denote by H^k the conventional Sobolev space of functions defined on Ω , and denote by H_0^1 the subspace of H^1 consisting of functions whose traces on the boundary are zero. Let $\mathbf{X}_h = (\dot{S}_h^2)^3$ and $V_h = S_h^1$ so that \mathbf{X}_h and V_h are finite element subspaces of $(H_0^1)^3$ and L^2 , respectively, with the following approximation property:

$$(2.1) \quad \inf_{\mathbf{v}_h \in \mathbf{X}_h} \|\mathbf{v} - \mathbf{v}_h\|_{L^q} \leq C \|\mathbf{v}\|_{H^{l+1}} h^{l+1+3/q-3/2}, \quad \forall \mathbf{v} \in (H_0^1 \cap H^l)^3, \\ \forall 0 \leq l \leq 2, \quad 2 \leq q \leq 6,$$

$$(2.2) \quad \inf_{q_h \in V_h} \|q - q_h\|_{L^2} \leq C \|q\|_{H^{l+1}} h^{l+1}, \quad \forall q \in H^l, \quad 0 \leq l \leq 1,$$

for some positive constant C which is independent of the mesh size h . It is well known that the following “inf-sup condition” [2, 4] is satisfied by the finite element spaces \mathbf{X}_h and V_h :

$$(2.3) \quad \|q_h\|_{L^2} \leq C \sup_{\substack{\mathbf{v}_h \in \mathbf{X}_h \\ \mathbf{v}_h \neq 0}} \frac{|(\nabla \cdot \mathbf{v}_h, q_h)|}{\|\mathbf{v}_h\|_{H^1}}, \quad \forall q_h \in V_h.$$

This condition guarantees the existence, uniqueness and stability of the finite element solution, but it will not be used explicitly in this paper. We remark that other conforming finite element spaces satisfying the three conditions above can also be used in the subsequent analysis.

Let the time interval $[0, T]$ be partitioned uniformly into $0 = t_0 < t_1 < \dots < t_N = T$, and denote $\tau = t_{m+1} - t_m$. For any sequence of functions g_0, g_1, \dots, g_N , we define

$$D_\tau g^{n+1} = \frac{g^{n+1} - g^n}{\tau}, \quad n = 1, 2, \dots, N-1.$$

For any given $\mathbf{u}_h^n \in \mathbf{X}_h$, we look for $\mathbf{u}_h^{n+1} \in \mathbf{X}_h$ and $p_h^{n+1} \in V_h$ as the solution of the following linearized finite element equations

$$(2.4) \quad \begin{aligned} & (D_\tau \mathbf{u}_h^{n+1}, \mathbf{v}_h) + \frac{1}{2}(\mathbf{u}_h^n \cdot \nabla \mathbf{u}_h^{n+1}, \mathbf{v}_h) - \frac{1}{2}(\mathbf{u}_h^{n+1}, \mathbf{u}_h^n \cdot \nabla \mathbf{v}_h) \\ & + \mu(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) - (p_h^{n+1}, \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{X}_h, \end{aligned}$$

$$(2.5) \quad (\nabla \cdot \mathbf{u}_h^{n+1}, q_h) = 0, \quad \forall q_h \in V_h,$$

where $\mathbf{u}_h^0 \in \mathbf{X}_h$ is simply the Lagrange interpolation of \mathbf{u}_0 .

Note that for any given initial data $\mathbf{u}_h^0 \in \mathbf{X}_h$ and mesh sizes $\tau > 0$, $h > 0$, the linearized equations (2.4)-(2.5) have a unique finite element solution \mathbf{u}_h^{n+1} , $n = 0, 1, \dots, N-1$, which satisfies the discrete energy estimate:

$$(2.6) \quad \max_{0 \leq n \leq N-1} \frac{1}{2} \|\mathbf{u}_h^{n+1}\|_{L^2}^2 + \sum_{n=0}^{N-1} \tau \|\nabla \mathbf{u}_h^{n+1}\|_{L^2}^2 \leq \frac{1}{2} \|\mathbf{u}_h^0\|_{L^2}^2.$$

This finite element scheme has been studied in [12], where error estimates of the finite element solution were presented by assuming that the Navier–Stokes equations have a smooth solution. Here we have a different purpose in using the numerical scheme (2.4)-(2.5). We define the numerical solution

$$(2.7) \quad \mathbf{u}_{h,\tau}(x, t) = \mathbf{u}_h^n(x) \quad \text{for } t \in (t_{n-1}, t_n] \text{ and } x \in \Omega,$$

and present our main result in the following theorem.

Theorem 2.1. *For any $M > 0$ there exist positive constants τ_M and h_M such that if a numerical solution $\mathbf{u}_{h,\tau}$ defined by (2.7) satisfies*

$$(2.8) \quad \|\mathbf{u}_{h,\tau}\|_{L^\infty(0,T;L^4)} + \|\mathbf{u}_0\|_{H_0^1 \cap H^2} + 1 \leq M,$$

and

$$(2.9) \quad \tau < \tau_M, \quad h < h_M,$$

then the Navier–Stokes equations (1.1)-(1.2) possess a unique solution satisfying

$$(2.10) \quad \mathbf{u} \in L^\infty(0, T; (H_0^1 \cap H^2)^3), \quad \partial_t \mathbf{u} \in L^2(0, T; (H^1)^3).$$

The positive constants τ_M and h_M are decreasing functions of M and independent of T .

Remark 2.1. Theorem 2.1 states that, when solving the Navier–Stokes equations, we do not need to assume existence, uniqueness or regularity of the solution. Instead, if we have a initial data \mathbf{u}_0 and a numerical solution $\mathbf{u}_{h,\tau}$, one can pick up M satisfying (2.8) and refine the mesh according to (2.9). If the conditions (2.8)-(2.9) are satisfied by one numerical solution, then one can say that, for the given initial data the Navier–Stokes equations admit a unique solution satisfying (2.10). Conversely, if the Navier–Stokes equations have a smooth solution satisfying (2.10), then there exists a positive constant M such that (2.9) implies (2.8).

Remark 2.2. From the proof of the theorem in the next section, one can see that the numerical solution successfully approximates the exact solution in the sense that

$$(2.11) \quad \|\mathbf{u}_{h,\tau} - \mathbf{u}\|_{L^\infty(0,T;L^2)}^2 \leq \tau + h^{3/2}.$$

Clearly, the order of this a posteriori error estimate can be improved. It is not our purpose to present optimal-order error estimates here.

Remark 2.3. In this paper, we only prove existence of such constants as τ_M and h_M . The $L^\infty(0, T; L^4)$ norm used in (2.8) may be replaced by some other norm. It is interesting to find useful expressions for τ_M and h_M in terms of M via more delicate analysis. Then Theorem 2.1 can be used in practical computations of the Navier–Stokes equations, and (2.11) can be viewed as an a posteriori error estimate.

Remark 2.4. It is possible to extend Theorem 2.1 to other nonlinear time-evolution equations for which global existence, uniqueness and regularity of the solution are unknown but local existence, uniqueness and regularity are known for smooth initial data. For such equations, our method proves global uniqueness and regularity of the solution as well as convergence of the numerical solutions in an a posteriori way. This can be viewed as an improvement of the traditional approach on error estimates of numerical solutions (which is based on global well-posedness assumptions that are not proved yet).

3. Proof of Theorem 2.1

It is well known that a solution with the regularity (2.10) is unique. It suffices to prove the regularity of the solution. In the rest part of this paper, we denote by C_{p_1, p_2, \dots, p_m} a generic positive constant which may depend on the parameters p_1, p_2, \dots, p_m and μ , but is independent of n, k, τ, h and T .

3.1. Local regularity estimates

In this subsection, we prove the following lemma, which is used in the next subsection to prove Theorem 2.1.

Lemma 3.1. *There exists a decreasing function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and an increasing function $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that if $\mathbf{u}_0 \in (H_0^1 \cap H^2)^3$ and the solution of (1.1)-(1.2) satisfies $\mathbf{u} \in L^\infty(0, T; L^4)$, then*

$$\begin{aligned} & \|\partial_{tt}\mathbf{u}\|_{L^2(0, T+\varphi(\|\mathbf{u}\|_{L^\infty(0, T; L^4)}+\|\mathbf{u}_0\|_{H^2}); H^{-1})} + \|\partial_t\mathbf{u}\|_{L^\infty(0, T+\varphi(\|\mathbf{u}\|_{L^\infty(0, T; L^4)}+\|\mathbf{u}_0\|_{H^2}); L^2)} \\ & + \|\partial_t\mathbf{u}\|_{L^2(0, T+\varphi(\|\mathbf{u}\|_{L^\infty(0, T; L^4)}+\|\mathbf{u}_0\|_{H^2}); H^1)} + \|\mathbf{u}\|_{L^2(0, T+\varphi(\|\mathbf{u}\|_{L^\infty(0, T; L^4)}+\|\mathbf{u}_0\|_{H^2}); H^2)} \\ & + \|\mathbf{u}\|_{L^\infty(0, T+\varphi(\|\mathbf{u}\|_{L^\infty(0, T; L^4)}+\|\mathbf{u}_0\|_{H^2}); H^2)} + \|p\|_{L^\infty(0, T+\varphi(\|\mathbf{u}\|_{L^\infty(0, T; L^4)}+\|\mathbf{u}_0\|_{H^2}); H^1)} \\ & + \|\partial_t p\|_{L^2(0, T+\varphi(\|\mathbf{u}\|_{L^\infty(0, T; L^4)}+\|\mathbf{u}_0\|_{H^2}); L^2)} \\ & \leq \Phi(\|\mathbf{u}\|_{L^\infty(0, T; L^4)} + \|\mathbf{u}_0\|_{H^2}), \end{aligned}$$

where the functions φ and Φ do not depend on T .

In order to prove Lemma 3.1, we introduce some lemmas below.

Lemma 3.2. *There exists a positive constant C_0 such that*

$$\begin{aligned} \|\mathbf{u}\|_{L^4} & \leq C_0 \|\mathbf{u}\|_{H^1}, \\ \|\mathbf{u}\|_{L^4} & \leq C_0 \|\mathbf{u}\|_{L^2}^{1/4} \|\nabla \mathbf{u}\|_{L^2}^{3/4}, \\ \|\nabla \mathbf{u}\|_{L^4} & \leq C_0 \|\nabla \mathbf{u}\|_{L^2}^{1/4} \|\Delta \mathbf{u}\|_{L^2}^{3/4}. \end{aligned}$$

Lemma 3.3. *There exists a decreasing function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that if $\mathbf{u}_0 \in (H_0^1 \cap H^2)^3$, then the solution of (1.1)-(1.2) satisfies*

$$\|\mathbf{u}\|_{L^\infty(0, \alpha(\|\mathbf{u}_0\|_{H^2}); H^1)} \leq \|\mathbf{u}_0\|_{H^1} + 1.$$

Lemma 3.4. *There exists an increasing function $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that if $\mathbf{u}_0 \in (H_0^1 \cap H^2)^3$ and the solution of (1.1)-(1.2) satisfies $\mathbf{u} \in L^\infty(0, T; L^4)$, then*

$$\begin{aligned} & \|\partial_{tt}\mathbf{u}\|_{L^2(0, T; H^{-1})} + \|\partial_t\mathbf{u}\|_{L^\infty(0, T; L^2)} + \|\partial_t\mathbf{u}\|_{L^2(0, T; H^1)} + \|\mathbf{u}\|_{L^2(0, T; H^2)} \\ & + \|\mathbf{u}\|_{L^\infty(0, T; H^2)} + \|p\|_{L^\infty(0, T; H^1)} + \|\partial_t p\|_{L^2(0, T; L^2)} \\ & \leq \beta(\|\mathbf{u}\|_{L^\infty(0, T; L^4)} + \|\mathbf{u}_0\|_{H^2}), \end{aligned}$$

where the function β does not depend on T .

Remark 3.1. Lemma 3.2 concerns a Sobolev embedding inequality and a well-known Sobolev interpolation inequality, which can be found in [1, 20, 28]. Based on the proof of Lemma 3.3 and Lemma 3.4 below, one can choose $\alpha(s) = \frac{1}{[C_1^* + C_1^*[C_0 + (C_0 + 1)s]^9]^2}$ and $\beta(s) = C_1 + C_1 s^{15}$, where C_1 and C_1^* are some positive constants. These expressions for α and β are not sharp and may be improved.

From Lemma 3.3 we see that

$$\|\mathbf{u}\|_{L^\infty(0, T + \alpha(\|\mathbf{u}\|_{L^\infty(0, T; H^2)}); H^1)} \leq \|\mathbf{u}\|_{L^\infty(0, T; H^1)} + 1,$$

and from Lemma 3.4 we see that

$$\|\mathbf{u}\|_{L^\infty(0, T; H^2)} \leq \beta(\|\mathbf{u}\|_{L^\infty(0, T; L^4)} + \|\mathbf{u}_0\|_{H^2}),$$

and

$$\begin{aligned} & \|\partial_{tt}\mathbf{u}\|_{L^2(0, T + \alpha(\|\mathbf{u}\|_{L^\infty(0, T; H^2)}); H^{-1})} + \|\partial_t\mathbf{u}\|_{L^\infty(0, T + \alpha(\|\mathbf{u}\|_{L^\infty(0, T; H^2)}); L^2)} \\ & + \|\partial_t\mathbf{u}\|_{L^2(0, T + \alpha(\|\mathbf{u}\|_{L^\infty(0, T; H^2)}); H^1)} + \|\mathbf{u}\|_{L^2(0, T + \alpha(\|\mathbf{u}\|_{L^\infty(0, T; H^2)}); H^2)} \\ & + \|\mathbf{u}\|_{L^\infty(0, T + \alpha(\|\mathbf{u}\|_{L^\infty(0, T; H^2)}); H^2)} + \|p\|_{L^\infty(0, T + \alpha(\|\mathbf{u}\|_{L^\infty(0, T; H^2)}); H^1)} \\ & + \|\partial_t p\|_{L^2(0, T + \alpha(\|\mathbf{u}\|_{L^\infty(0, T; H^2)}); L^2)} \\ & \leq \beta(C_0 \|\mathbf{u}\|_{L^\infty(0, T + \alpha(\|\mathbf{u}\|_{L^\infty(0, T; H^2)}); H^1)} + \|\mathbf{u}_0\|_{H^2}). \end{aligned}$$

The last three inequalities imply Lemma 3.1 with

$$\varphi(s) = \alpha(\beta(s)) \quad \text{and} \quad \Phi(s) = \beta(C_0 \beta(s) + s + C_0).$$

It remains to prove Lemma 3.3 and Lemma 3.4.

Proof of Lemma 3.4. It is well known that if $\mathbf{u}_0 \in H_0^1 \cap H^2$ and $\mathbf{u} \in L^\infty(0, T; L^4)$, then the Navier–Stokes equations have a strong solution. For the convenience of the readers, we present a priori estimates here, where all the positive constants are independent of T .

Integrating (1.1) against $\partial_t \mathbf{u}$, we obtain

$$\|\partial_t \mathbf{u}\|_{L^2}^2 + \frac{d}{dt} \left(\frac{\mu}{2} \|\nabla \mathbf{u}\|_{L^2}^2 \right) \leq \frac{1}{2} \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2}^2 + \frac{1}{2} \|\partial_t \mathbf{u}\|_{L^2}^2,$$

which reduces to

$$\|\partial_t \mathbf{u}\|_{L^2}^2 + \frac{d}{dt} \left(\mu \|\nabla \mathbf{u}\|_{L^2}^2 \right) \leq \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2}^2,$$

and from (1.1) we further derive

$$\begin{aligned} \mu^2 \|\Delta \mathbf{u}\|_{L^2}^2 &\leq 2 \|\partial_t \mathbf{u}\|_{L^2}^2 + 2 \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2}^2 \\ &\leq 4 \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2}^2 - \frac{d}{dt} \left(2\mu \|\nabla \mathbf{u}\|_{L^2}^2 \right). \end{aligned}$$

The sum of the last two inequalities gives

$$\begin{aligned} &\|\partial_t \mathbf{u}\|_{L^2}^2 + \mu^2 \|\Delta \mathbf{u}\|_{L^2}^2 + \frac{d}{dt} \left(3\mu \|\nabla \mathbf{u}\|_{L^2}^2 \right) \\ &\leq 5 \|\mathbf{u}\|_{L^4}^2 \|\nabla \mathbf{u}\|_{L^4}^2 \\ &\leq C \|\mathbf{u}\|_{L^4}^2 \|\nabla \mathbf{u}\|_{L^2}^{1/2} \|\Delta \mathbf{u}\|_{L^2}^{3/2} \\ &\leq C \|\mathbf{u}\|_{L^4}^8 \|\nabla \mathbf{u}\|_{L^2}^2 + \frac{\mu^2}{2} \|\Delta \mathbf{u}\|_{L^2}^2. \end{aligned}$$

which reduces to

$$\begin{aligned} &\|\partial_t \mathbf{u}\|_{L^2(0,T;L^2)}^2 + \frac{\mu^2}{2} \|\Delta \mathbf{u}\|_{L^2(0,T;L^2)}^2 + 3\mu \|\nabla \mathbf{u}\|_{L^\infty(0,T;L^2)}^2 \\ &\leq 3\mu \|\nabla \mathbf{u}_0\|_{L^2}^2 + C \|\mathbf{u}\|_{L^\infty(0,T;L^4)}^8 \|\nabla \mathbf{u}\|_{L^2(0,T;L^2)}^2 \\ &\leq 3\mu \|\nabla \mathbf{u}_0\|_{L^2}^2 + C \|\mathbf{u}\|_{L^\infty(0,T;L^4)}^8 \|\mathbf{u}_0\|_{L^2}^2 \\ (3.1) \quad &\leq C + C(\|\mathbf{u}\|_{L^\infty(0,T;L^4)} + \|\mathbf{u}_0\|_{H^2})^{10}. \end{aligned}$$

Differentiating (1.1) with respect to t , we obtain

$$(3.2) \quad \partial_{tt} \mathbf{u} - \mu \Delta \partial_t \mathbf{u} + \nabla \partial_t p = -\nabla \cdot (\partial_t \mathbf{u} \otimes \mathbf{u}) - \nabla \cdot (\mathbf{u} \otimes \partial_t \mathbf{u}).$$

Integrating the equation above against $\partial_t \mathbf{u}$, we obtain

$$\frac{d}{dt} \left(\frac{1}{2} \|\partial_t \mathbf{u}\|_{L^2}^2 \right) + \frac{\mu}{2} \|\nabla \partial_t \mathbf{u}\|_{L^2}^2 \leq \frac{1}{\mu} \|\partial_t \mathbf{u} \otimes \mathbf{u}\|_{L^2}^2 + \frac{1}{\mu} \|\mathbf{u} \otimes \partial_t \mathbf{u}\|_{L^2}^2,$$

and from (3.2) we further derive that

$$\begin{aligned} \|\partial_{tt} \mathbf{u}\|_{H^{-1}}^2 &\leq 2\mu^2 \|\nabla \partial_t \mathbf{u}\|_{L^2}^2 + 4 \|\partial_t \mathbf{u} \otimes \mathbf{u}\|_{L^2}^2 \\ &\leq 12 \|\partial_t \mathbf{u} \otimes \mathbf{u}\|_{L^2}^2 - \frac{d}{dt} (2\mu \|\partial_t \mathbf{u}\|_{L^2}^2). \end{aligned}$$

The last two inequalities imply

$$\begin{aligned} &\|\partial_{tt} \mathbf{u}\|_{H^{-1}}^2 + \frac{d}{dt} (4\mu \|\partial_t \mathbf{u}\|_{L^2}^2) + 2\mu^2 \|\nabla \partial_t \mathbf{u}\|_{L^2}^2 \\ &\leq 20 \|\partial_t \mathbf{u} \otimes \mathbf{u}\|_{L^2}^2 \\ &\leq 20 \|\partial_t \mathbf{u}\|_{L^4}^2 \|\mathbf{u}\|_{L^4}^2 \\ &\leq 20 \|\partial_t \mathbf{u}\|_{L^2}^{1/2} \|\partial_t \mathbf{u}\|_{L^6}^{3/2} \|\mathbf{u}\|_{L^4}^2 \\ &\leq C \|\partial_t \mathbf{u}\|_{L^2}^{1/2} \|\nabla \partial_t \mathbf{u}\|_{L^2}^{3/2} \|\mathbf{u}\|_{L^4}^2 \end{aligned}$$

$$\leq C\|\mathbf{u}\|_{L^4}^8\|\partial_t\mathbf{u}\|_{L^2}^2 + \mu^2\|\nabla\partial_t\mathbf{u}\|_{L^2}^2,$$

which further reduces to

$$\begin{aligned} & \|\partial_{tt}\mathbf{u}\|_{L^2(0,T;H^{-1})}^2 + 4\mu\|\partial_t\mathbf{u}\|_{L^\infty(0,T;L^2)}^2 + \mu^2\|\nabla\partial_t\mathbf{u}\|_{L^2(0,T;L^2)}^2 \\ & \leq 4\mu\|\partial_t\mathbf{u}_0\|_{L^2}^2 + C\|\mathbf{u}\|_{L^\infty(0,T;L^4)}^8\|\partial_t\mathbf{u}\|_{L^2(0,T;L^2)}^2 \\ & \leq 4\mu\|\mathbf{u}_0 \cdot \nabla\mathbf{u}_0 - \mu\Delta\mathbf{u}_0\|_{L^2}^2 + C\|\mathbf{u}\|_{L^\infty(0,T;L^4)}^8[C + C(\|\mathbf{u}\|_{L^\infty(0,T;L^4)} + \|\mathbf{u}_0\|_{H^2})]^{10} \\ (3.3) \quad & \leq C + C(\|\mathbf{u}\|_{L^\infty(0,T;L^4)} + \|\mathbf{u}_0\|_{H^2})^{18} \end{aligned}$$

where we have used (3.1) in estimating $\|\partial_t\mathbf{u}\|_{L^2(0,T;L^2)}^2$. Substituting the estimate above into (3.2), we further derive

$$\begin{aligned} \|\partial_t p\|_{L^2(0,T;L^2)} & \leq C\|\partial_{tt}\mathbf{u}\|_{L^2(0,T;H^{-1})} + C\|\nabla\partial_t\mathbf{u}\|_{L^2(0,T;L^2)} + C\|\partial_t\mathbf{u} \otimes \mathbf{u}\|_{L^2} \\ (3.4) \quad & \leq C + C(\|\mathbf{u}\|_{L^\infty(0,T;L^4)} + \|\mathbf{u}_0\|_{H^2})^9. \end{aligned}$$

From (1.1) we see that

$$\begin{aligned} & \|\mathbf{u}\|_{L^\infty(0,T;H^2)} \\ & \leq C\|\Delta\mathbf{u}\|_{L^\infty(0,T;L^2)} \\ & \leq C\|\partial_t\mathbf{u}\|_{L^\infty(0,T;L^2)} + C\|\mathbf{u} \cdot \nabla\mathbf{u}\|_{L^\infty(0,T;L^2)} \\ & \leq C\|\partial_t\mathbf{u}\|_{L^\infty(0,T;L^2)} + \|\mathbf{u}\|_{L^\infty(0,T;L^\infty)}\|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2)} \\ & \leq C\|\partial_t\mathbf{u}\|_{L^\infty(0,T;L^2)} + \|\mathbf{u}\|_{L^\infty(0,T;L^\infty)}\|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2)} \\ & \leq C\|\partial_t\mathbf{u}\|_{L^\infty(0,T;L^2)} + C\|\mathbf{u}\|_{L^\infty(0,T;H^1)}^{1/2}\|\mathbf{u}\|_{L^\infty(0,T;H^2)}^{1/2}\|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2)} \\ & \leq C\|\partial_t\mathbf{u}\|_{L^\infty(0,T;L^2)} + C\|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2)}^3 + \frac{1}{2}\|\mathbf{u}\|_{L^\infty(0,T;H^2)}, \end{aligned}$$

which implies

$$\begin{aligned} \|\mathbf{u}\|_{L^\infty(0,T;H^2)} & \leq C\|\partial_t\mathbf{u}\|_{L^\infty(0,T;L^2)} + C\|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2)}^3 \\ (3.5) \quad & \leq [C + C(\|\mathbf{u}\|_{L^\infty(0,T;L^4)} + \|\mathbf{u}_0\|_{H^2})^{15}]. \end{aligned}$$

With the above estimates of $\|\partial_t\mathbf{u}\|_{L^\infty(0,T;L^2)}$, $\|\partial_t\mathbf{u} \otimes \mathbf{u}\|_{L^\infty(0,T;L^2)}$ and $\|\mathbf{u}\|_{L^\infty(0,T;H^2)}$, from (1.1) we also derive

$$(3.6) \quad \|p\|_{L^\infty(0,T;H^1)} \leq [C + C(\|\mathbf{u}\|_{L^\infty(0,T;L^4)} + \|\mathbf{u}_0\|_{H^2})^{15}].$$

The inequalities (3.1) and (3.3)-(3.6) imply Lemma 3.4 with

$$(3.7) \quad \beta(s) = C_1 + C_1 s^{15}$$

where C_1 is some positive constant which is independent of T . \square

Proof of Lemma 3.3. It is well known that the Navier-Stokes equations admit local H^2 solutions for a given H^2 initial data. If we let

$$t_* = \sup\{s : \|\mathbf{u}\|_{L^\infty(0,s;H^1)} \leq \|\mathbf{u}_0\|_{H^1} + 1\},$$

then from (3.3) we know that $\mathbf{u} \in C([0, t_*]; H^1)$ and so $\|\mathbf{u}_0\|_{H^1} + 1 = \|\mathbf{u}\|_{L^\infty(0, t_*; H^1)}$. From (3.3) we see that

$$\|\mathbf{u}_0\|_{H^1} + 1 = \|\mathbf{u}\|_{L^\infty(0, t_*; H^1)} \leq \|\mathbf{u}_0\|_{H^1} + \|\partial_t\mathbf{u}\|_{L^2(0, t_*; H^1)} t_*^{1/2}$$

$$\begin{aligned}
&\leq \|\mathbf{u}_0\|_{H^1} + [C_1^* + C_1^*(\|\mathbf{u}\|_{L^\infty(0,t_*;L^4)} + \|\mathbf{u}_0\|_{H^2})^9] t_*^{1/2} \\
&\leq \|\mathbf{u}_0\|_{H^1} + [C_1^* + C_1^*(C_0\|\mathbf{u}\|_{L^\infty(0,t_*;H^1)} + \|\mathbf{u}_0\|_{H^2})^9] t_*^{1/2},
\end{aligned}$$

which implies

$$t_* \geq \frac{1}{[C_1^* + C_1^*(C_0 + C_0\|\mathbf{u}_0\|_{H^1} + \|\mathbf{u}_0\|_{H^2})^9]^2} \geq \frac{1}{[C_1^* + C_1^*[C_0 + (C_0 + 1)\|\mathbf{u}_0\|_{H^2}]^9]^2}.$$

Therefore, Lemma 3.3 holds with $\alpha(s) = \frac{1}{[C_1^* + C_1^*[C_0 + (C_0 + 1)s]^9]^2}$. \square

3.2. Global regularity estimates based on the numerical solution

We introduce the Stokes Ritz projection operator $(R_h, P_h) : (H_0^1)^3 \times L^2 \rightarrow \mathbf{X}_h \times V_h$ by

$$(3.8) \quad (\nabla(\mathbf{w} - R_h(\mathbf{w}, p)), \nabla \mathbf{v}_h) - (p - P_h(\mathbf{w}, p), \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{X}_h,$$

$$(3.9) \quad (\nabla \cdot R_h(\mathbf{w}, p), q_h) = 0, \quad \forall q_h \in V_h,$$

and impose the condition $\int_\Omega (p - P_h(\mathbf{w}, p)) \, dx = 0$ for uniqueness. This linear projection operator has the approximation property

$$\begin{aligned}
&h^{3/2-3/q} \|\mathbf{w} - R_h(\mathbf{w}, p)\|_{L^q} + h \|\mathbf{w} - R_h(\mathbf{w}, p)\|_{H^1} + h \|p - P_h(\mathbf{w}, p)\|_{L^2} \\
(3.10) \quad &\leq Ch^{l+1} (\|\mathbf{w}\|_{H^{l+1}} + \|p\|_{H^l}), \quad l = 0, 1, \quad 2 \leq q \leq 6,
\end{aligned}$$

for any divergence-free vector field $\mathbf{w} \in (H_0^1 \cap H^2)^3$ and scalar field $p \in H^1$; see [29] for the proof of the case $q = 2$; the case $2 < q \leq 6$ can be proved by using the inverse inequality and the Bramble–Hilbert lemma. This approximation property, together with the inverse inequality

$$(3.11) \quad \|\mathbf{v}_h\|_{W^{1,q_2}} \leq Ch^{3/q_2-3/q_1+l-1} \|\mathbf{v}_h\|_{W^{l,q_1}}, \quad \forall \mathbf{v}_h \in \mathbf{X}_h, \quad 1 \leq q_1 \leq q_2 \leq \infty, \quad l = 0, 1,$$

will be used in the analysis, and we also need the discrete Gronwall's inequality [13]:

Lemma 3.5. *Let τ , B and a_m , b_m , c_m , γ_m , for integers $k \geq 0$, be nonnegative numbers such that*

$$a_{n+1} + \tau \sum_{m=0}^n b_{m+1} \leq \tau \sum_{m=0}^n \gamma_{m+1} a_{m+1} + \tau \sum_{m=0}^n c_{m+1} + B, \quad \text{for } n \geq 0,$$

and suppose that $\tau \gamma_{m+1} < 1/2$ for all $m \geq 0$. Then

$$a_{n+1} + \tau \sum_{m=0}^n b_{m+1} \leq \exp \left(2 \sum_{m=0}^n \tau \gamma_{m+1} \right) \left(\tau \sum_{m=0}^n c_{m+1} + B \right), \quad \text{for } n \geq 0.$$

To prove Theorem 2.1, we use mathematical induction on

$$(3.12) \quad \|\mathbf{u}_{h,\tau} - \mathbf{u}\|_{L^\infty(0,t_k;L^4)} \leq 1.$$

Since \mathbf{u}_h^0 is the Lagrange interpolation of \mathbf{u}_0 , we have $\|\mathbf{u}_h^0 - \mathbf{u}_0\|_{L^4} \leq C_2 \|\mathbf{u}_0\|_{H^2} h^{5/4}$ for some positive constant C_2 . Thus our assumption holds for $k = 0$ when $h < (C_2 \|\mathbf{u}_0\|_{H^2})^{-4/5}$. We shall prove that if (3.12) holds for some nonnegative integer k , then it also holds when t_k is replaced by t_{k+1} .

To simplify the notations, we define $M = \|\mathbf{u}_{h,\tau}\|_{L^\infty(0,T;L^4)} + 1 + \|\mathbf{u}_0\|_{H^2}$. From the induction assumption (3.12) we see that $\|\mathbf{u}\|_{L^\infty(0,t_k;L^4)} \leq \|\mathbf{u}_{h,\tau}\|_{L^\infty(0,t_k;L^4)} + 1$. Therefore, when

$$(3.13) \quad \tau < \varphi(M)/2$$

Lemma 3.1 implies

$$(3.14) \quad \begin{aligned} & \|\partial_{tt}\mathbf{u}\|_{L^2(0,t_{k+2};H^{-1})} + \|\partial_t\mathbf{u}\|_{L^\infty(0,t_{k+2};L^2)} + \|\partial_t\mathbf{u}\|_{L^2(0,t_{k+2};H^1)} + \|\mathbf{u}\|_{L^2(0,t_{k+2};H^2)} \\ & + \|\mathbf{u}\|_{L^\infty(0,t_{k+2};H^2)} + \|\partial_t p\|_{L^2(0,t_{k+2};H^2)} + \|p\|_{L^\infty(0,t_{k+2};H^1)} \\ & \leq \Phi(\|\mathbf{u}\|_{L^\infty(0,t_k;L^4)} + \|\mathbf{u}_0\|_{H^2}) \\ & \leq \Phi(M). \end{aligned}$$

Under this regularity, the solution \mathbf{u} satisfies the variational equations

$$(3.15) \quad \begin{aligned} & (D_\tau \mathbf{u}^{n+1}, \mathbf{v}_h) + \frac{1}{2}(\mathbf{u}^n \cdot \nabla \mathbf{u}^{n+1}, \mathbf{v}_h) - \frac{1}{2}(\mathbf{u}^{n+1}, \mathbf{u}^n \cdot \nabla \mathbf{v}_h) \\ & + \mu(\nabla \mathbf{u}^{n+1}, \nabla \mathbf{v}_h) - (p^{n+1}, \nabla \cdot \mathbf{v}_h) = (\mathbf{E}^{n+1}, \mathbf{v}_h) + (\mathbf{F}^{n+1}, \mathbf{v}_h), \end{aligned}$$

$$(3.16) \quad (\nabla \cdot \mathbf{u}^{n+1}, q_h) = 0,$$

for $n = 0, 1, \dots, k$, where

$$(3.17) \quad \mathbf{E}^{n+1} = D_\tau \mathbf{u}^{n+1} - \partial_t \mathbf{u}^{n+1} + \nabla \cdot \left[\frac{1}{2}(\mathbf{u}^n - \mathbf{u}^{n+1}) \otimes \mathbf{u}^{n+1} \right],$$

$$(3.18) \quad \mathbf{F}^{n+1} = \frac{1}{2} \mathbf{u}^{n+1} \otimes (\mathbf{u}^{n+1} - \mathbf{u}^n),$$

are the truncation errors due to the time discretization, satisfying

$$(3.19) \quad \begin{aligned} & \left(\sum_{n=0}^k \tau |\mathbf{E}^{n+1}|_{H^{-1}}^2 \right)^{\frac{1}{2}} + \left(\sum_{n=0}^k \tau |\mathbf{F}^{n+1}|_{L^2}^2 \right)^{\frac{1}{2}} \\ & \leq C\tau \|\partial_{tt}\mathbf{u}\|_{L^2(0,t_{k+1};H^{-1})} + C\tau \|\partial_t \mathbf{u}\|_{L^2(0,t_{k+1};L^4)} \|\mathbf{u}\|_{L^\infty(0,t_{k+1};L^4)} \\ & \leq C(\Phi(M) + \Phi(M)^2)\tau \\ & \leq C\Phi(M)^2\tau. \end{aligned}$$

Let $\mathbf{e}_h^{n+1} := \mathbf{u}_h^{n+1} - R_h(\mathbf{u}^{n+1}, p^{n+1})$ and $\eta_h^{n+1} := p_h^{n+1} - P_h(\mathbf{u}^{n+1}, p^{n+1})$. Then the difference between (2.4)-(2.5) and (3.15)-(3.16) gives

$$\begin{aligned} & (D_\tau \mathbf{e}_h^{n+1}, \mathbf{v}_h) + \frac{1}{2}(\mathbf{u}_h^n \cdot \nabla \mathbf{e}_h^{n+1}, \mathbf{v}_h) - \frac{1}{2}(\mathbf{e}_h^{n+1}, \mathbf{u}_h^n \cdot \nabla \mathbf{v}_h) \\ & + \frac{1}{2}(\mathbf{e}_h^n \cdot \nabla \mathbf{u}^{n+1}, \mathbf{v}_h) - \frac{1}{2}(\mathbf{u}^{n+1}, \mathbf{e}_h^n \cdot \nabla \mathbf{v}_h) \\ & + \mu(\nabla \mathbf{e}_h^{n+1}, \nabla \mathbf{v}_h) - (\eta_h^{n+1}, \nabla \cdot \mathbf{v}_h) \\ & = (D_\tau \mathbf{u}^{n+1} - R_h(D_\tau \mathbf{u}^{n+1}, D_\tau p^{n+1}), \mathbf{v}_h) \\ & + (\mathbf{E}^{n+1}, \mathbf{v}_h) + (\mathbf{F}^{n+1}, \nabla \mathbf{v}_h) \\ & + \frac{1}{2}(\mathbf{u}_h^n \cdot \nabla (\mathbf{u}^{n+1} - R_h(\mathbf{u}^{n+1}, p^{n+1})), \mathbf{v}_h) \\ & + \frac{1}{2}((\mathbf{u}^n - R_h(\mathbf{u}^n, p^n)) \cdot \nabla \mathbf{u}^{n+1}, \mathbf{v}_h) \end{aligned}$$

$$(3.20) \quad \begin{aligned} & -\frac{1}{2}(\mathbf{u}^{n+1}, (\mathbf{u}^n - R_h(\mathbf{u}^n, p^n)) \cdot \nabla \mathbf{v}_h) \\ & -\frac{1}{2}(\mathbf{u}^{n+1} - R_h(\mathbf{u}^{n+1}, p^{n+1}), \mathbf{u}_h^n \cdot \nabla \mathbf{v}_h), \end{aligned}$$

$$(3.21) \quad (\nabla \cdot \mathbf{e}_h^{n+1}, q_h) = 0,$$

Substituting $\mathbf{v}_h = \mathbf{e}_h^{n+1}$ into the equation and using the identities

$$\begin{aligned} & (\nabla \cdot \mathbf{e}_h^{n+1}, \eta_h^{n+1}) = 0, \\ & \frac{1}{2}(\mathbf{e}_h^n \cdot \nabla \mathbf{u}^{n+1}, \mathbf{v}_h) - \frac{1}{2}(\mathbf{u}^{n+1}, \mathbf{e}_h^n \cdot \nabla \mathbf{v}_h) = -\frac{1}{2}((\nabla \cdot \mathbf{e}_h^n) \mathbf{u}^{n+1}, \mathbf{v}_h) - (\mathbf{u}^{n+1}, \mathbf{e}_h^n \cdot \nabla \mathbf{v}_h), \end{aligned}$$

we obtain

$$\begin{aligned} & D_\tau \left(\frac{1}{2} \|\mathbf{e}_h^{n+1}\|_{L^2}^2 \right) + \mu \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 \\ & = (D_\tau \mathbf{u}^{n+1} - R_h(D_\tau \mathbf{u}^{n+1}, D_\tau p^{n+1}), \mathbf{e}_h^{n+1}) \\ & \quad + (\mathbf{E}^{n+1}, \mathbf{e}_h^{n+1}) + (\mathbf{F}^{n+1}, \nabla \mathbf{e}_h^{n+1}) \\ & \quad + \frac{1}{2}(\mathbf{u}_h^n \cdot \nabla (\mathbf{u}^{n+1} - R_h(\mathbf{u}^{n+1}, p^{n+1})), \mathbf{e}_h^{n+1}) \\ & \quad + \frac{1}{2}((\mathbf{u}^n - R_h(\mathbf{u}^n, p^n)) \cdot \nabla \mathbf{u}^{n+1}, \mathbf{e}_h^{n+1}) \\ & \quad - \frac{1}{2}(\mathbf{u}^{n+1}, (\mathbf{u}^n - R_h(\mathbf{u}^n, p^n)) \cdot \nabla \mathbf{e}_h^{n+1}) \\ & \quad - \frac{1}{2}(\mathbf{u}^{n+1} - R_h(\mathbf{u}^{n+1}, p^{n+1}), \mathbf{u}_h^n \cdot \nabla \mathbf{e}_h^{n+1}) \\ & \quad + \frac{1}{2}((\nabla \cdot \mathbf{e}_h^n) \mathbf{u}^{n+1}, \mathbf{v}_h) + (\mathbf{u}^{n+1}, \mathbf{e}_h^n \cdot \nabla \mathbf{v}_h) \\ & \leq \|D_\tau \mathbf{u}^{n+1} - R_h(D_\tau \mathbf{u}^{n+1}, D_\tau p^{n+1})\|_{L^2} \|\mathbf{e}_h^{n+1}\|_{L^2} \\ & \quad + (\|\mathbf{E}^{n+1}\|_{H^{-1}} + C\|\mathbf{F}^{n+1}\|_{L^2}) \|\mathbf{e}_h^{n+1}\|_{H^1} \\ & \quad + \frac{1}{2} \|\mathbf{u}_h^n\|_{L^3} \|\nabla (\mathbf{u}^{n+1} - R_h(\mathbf{u}^{n+1}, p^{n+1}))\|_{L^2} \|\mathbf{e}_h^{n+1}\|_{L^6} \\ & \quad + \frac{1}{2} \|\mathbf{u}^n - R_h(\mathbf{u}^n, p^n)\|_{L^3} \|\nabla \mathbf{u}^{n+1}\|_{L^2} \|\mathbf{e}_h^{n+1}\|_{L^6} \\ & \quad + \frac{1}{2} \|\mathbf{u}^{n+1}\|_{L^4} \|\mathbf{u}^n - R_h(\mathbf{u}^n, p^n)\|_{L^4} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2} \\ & \quad + \frac{1}{2} \|\mathbf{u}^{n+1} - R_h(\mathbf{u}^{n+1}, p^{n+1})\|_{L^4} \|\mathbf{u}_h^n\|_{L^4} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2} \\ & \quad + \frac{1}{2} \|\nabla \cdot \mathbf{e}_h^n\|_{L^2} \|\mathbf{u}^{n+1}\|_{L^4} \|\mathbf{e}_h^{n+1}\|_{L^4} + \|\mathbf{u}^{n+1}\|_{L^4} \|\mathbf{e}_h^n\|_{L^4} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2} \\ & \leq C(\|D_\tau \mathbf{u}^{n+1}\|_{H^1} + \|D_\tau p^{n+1}\|_{L^2}) \|\nabla \mathbf{e}_h^{n+1}\|_{L^2} h \\ & \quad + C(\|\mathbf{E}^{n+1}\|_{H^{-1}} + \|\mathbf{F}^{n+1}\|_{L^2}) \|\nabla \mathbf{e}_h^{n+1}\|_{L^2} \\ & \quad + Ch \|\mathbf{u}_h^n\|_{L^3} (\|\mathbf{u}^{n+1}\|_{H^2} + \|p^{n+1}\|_{H^1}) \|\nabla \mathbf{e}_h^{n+1}\|_{L^2} \\ & \quad + Ch^{3/2} (\|\mathbf{u}^n\|_{H^2} + \|p^n\|_{H^1}) \|\nabla \mathbf{u}^{n+1}\|_{L^2} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2} \end{aligned}$$

$$\begin{aligned}
& + Ch^{5/4} \|\mathbf{u}^{n+1}\|_{L^4} (\|\mathbf{u}^n\|_{H^2} + \|p^n\|_{H^1}) \|\nabla \mathbf{e}_h^{n+1}\|_{L^2} \\
& + Ch^{5/4} (\|\mathbf{u}^{n+1}\|_{H^2} + \|p^{n+1}\|_{H^1}) \|\mathbf{u}_h^n\|_{L^4} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2} \\
& + C \|\nabla \mathbf{e}_h^n\|_{L^2} \|\mathbf{u}^{n+1}\|_{L^4} \|\mathbf{e}_h^{n+1}\|_{L^2}^{1/4} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^{3/4} \\
& + C \|\mathbf{u}^{n+1}\|_{L^4} \|\mathbf{e}_h^n\|_{L^2}^{1/4} \|\nabla \mathbf{e}_h^n\|_{L^2}^{3/4} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2} \\
& \leq C (\|D_\tau \mathbf{u}^{n+1}\|_{H^1}^2 + \|D_\tau p^{n+1}\|_{L^2}^2) h^2 + \frac{\mu}{16} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 \\
& + C (\|\mathbf{E}^{n+1}\|_{H^{-1}}^2 + \|\mathbf{F}^{n+1}\|_{L^2}^2) + \frac{\mu}{16} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 \\
& + C (h^2 \|\mathbf{u}_h^n\|_{L^3}^2 + \|\nabla \mathbf{u}^{n+1}\|_{L^2}^2 h^3) \sum_{m=n}^{n+1} (\|\mathbf{u}^m\|_{H^2}^2 + \|p^m\|_{H^1}^2) + \frac{\mu}{8} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 \\
& + C (h^{5/2} \|\mathbf{u}^{n+1}\|_{L^4}^2 + h^{5/2} \|\mathbf{u}_h^n\|_{L^4}^2) \sum_{m=n}^{n+1} (\|\mathbf{u}^m\|_{H^2}^2 + \|p^m\|_{H^1}^2) + \frac{\mu}{8} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 \\
& + C \|\mathbf{u}^{n+1}\|_{L^4}^8 (\|\mathbf{e}_h^{n+1}\|_{L^2}^2 + \|\mathbf{e}_h^n\|_{L^2}^2) + \frac{\mu}{16} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 + \frac{\mu}{16} \|\nabla \mathbf{e}_h^n\|_{L^2}^2 \\
& \leq C_2 h^2 (\|D_\tau \mathbf{u}^{n+1}\|_{H^1}^2 + \|D_\tau p^{n+1}\|_{L^2}^2) + C_2 \|\mathbf{E}^{n+1}\|_{H^{-1}}^2 + C_2 \|\mathbf{F}^{n+1}\|_{L^2}^2 \\
& + C_2 h^2 (\|\mathbf{u}_h^{n+1}\|_{L^3}^2 + \|\nabla \mathbf{u}^{n+1}\|_{L^2}^2 h) \sum_{m=n}^{n+1} (\|\mathbf{u}^m\|_{H^2}^2 + \|p^m\|_{H^1}^2) \\
& + C_2 h^2 (\|\mathbf{u}^{n+1}\|_{L^4}^2 h^{1/2} + \|\mathbf{u}_h^n\|_{L^4}^2 h^{1/2}) \sum_{m=n}^{n+1} (\|\mathbf{u}^m\|_{H^2}^2 + \|p^m\|_{H^1}^2) \\
(3.22) \quad & + C_2 \|\mathbf{u}^{n+1}\|_{L^4}^8 (\|\mathbf{e}_h^{n+1}\|_{L^2}^2 + \|\mathbf{e}_h^n\|_{L^2}^2) + \frac{7\mu}{16} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 + \frac{\mu}{16} \|\nabla \mathbf{e}_h^n\|_{L^2}^2.
\end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{n=0}^k \tau (\|D_\tau \mathbf{u}^{n+1}\|_{H^1}^2 + \|D_\tau p^{n+1}\|_{L^2}^2) \leq C (\|\partial_t \mathbf{u}\|_{L^2(0, t_{k+1}; H^1)}^2 + \|\partial_t p\|_{L^2(0, t_{k+1}; L^2)}^2) \leq C \Phi(M)^2, \\
& \sum_{n=0}^k \tau |\mathbf{E}^{n+1}|_{H^{-1}}^2 + \sum_{n=0}^k \tau |\mathbf{F}^{n+1}|_{L^2}^2 \leq C \Phi(M)^4 \tau^2, \\
& \sum_{n=0}^k \tau (\|\mathbf{u}_h^{n+1}\|_{L^3}^2 + \|\nabla \mathbf{u}^{n+1}\|_{L^2}^2 h + \|\mathbf{u}^{n+1}\|_{L^4}^2 h^{1/2} + \|\mathbf{u}_h^n\|_{L^4}^2 h^{1/2}) \sum_{m=n}^{n+1} (\|\mathbf{u}^m\|_{H^2}^2 + \|p^m\|_{H^1}^2) \\
& \leq \left[\tau \|\mathbf{u}_h^0\|_{L^4}^2 h^{1/2} + C \sum_{n=0}^k \tau (\|\nabla \mathbf{u}_h^{n+1}\|_{L^2}^2 + \|\nabla \mathbf{u}^{n+1}\|_{L^2}^2) \right] (\|\mathbf{u}\|_{L^\infty(0, t_{k+1}; H^2)}^2 + \|p\|_{L^\infty(0, t_{k+1}; H^1)}^2) \\
& \leq \left[C \|\mathbf{u}_0\|_{H^2}^2 + C \sum_{n=0}^k \tau \|\nabla \mathbf{u}^{n+1}\|_{L^2}^2 \right] \Phi(M)^2 \\
& \leq \left[C \|\mathbf{u}_0\|_{H^2}^2 + C \|\nabla \mathbf{u}\|_{L^2(0, t_{k+1}; L^2)}^2 + C \tau^2 \|\partial_t \nabla \mathbf{u}\|_{L^2(0, t_{k+1}; L^2)}^2 \right] \Phi(M)^2
\end{aligned}$$

$$\leq C\Phi(M)^4,$$

where we have used (2.6) in estimating $\sum_{n=0}^k \tau \|\nabla \mathbf{u}_h^{n+1}\|_{L^2}^2$ and used the expression $\nabla \mathbf{u}^{n+1} = \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \nabla \mathbf{u}(t) dt + \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \partial_t \nabla \mathbf{u}(s) ds$ in estimating $\sum_{n=0}^k \tau \|\nabla \mathbf{u}^{n+1}\|_{L^2}^2$. By using the last three inequalities and summing up (3.22) for $n = 0, 1, \dots, m$ (with $0 \leq m \leq k$), we obtain

$$\begin{aligned}
& \frac{1}{2} \|\mathbf{e}_h^{m+1}\|_{L^2}^2 + \frac{\mu}{2} \sum_{n=0}^m \tau \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 \\
& \leq \left(\frac{1}{2} + C_2 \|\mathbf{u}^1\|_{L^4}^8 \tau \right) \|\mathbf{e}_h^0\|_{L^2}^2 + \frac{\tau \mu}{16} \|\nabla \mathbf{e}_h^0\|_{L^2}^2 + C\Phi(M)^4(\tau^2 + h^2) \\
& \quad + 2C_3\Phi(M)^6 \sum_{n=0}^m \tau (\|\mathbf{u}^{n+1}\|_{L^4}^2 + \|\mathbf{u}^{n+2}\|_{L^4}^2) \|\mathbf{e}_h^{n+1}\|_{L^2}^2 \\
(3.23) \quad & \leq C\Phi(M)^4(\tau^2 + h^2) + 2C_3\Phi(M)^6 \sum_{n=0}^m \tau (\|\mathbf{u}^{n+1}\|_{L^4}^2 + \|\mathbf{u}^{n+2}\|_{L^4}^2) \|\mathbf{e}_h^{n+1}\|_{L^2}^2.
\end{aligned}$$

By using the expression $\mathbf{u}^{n+1} = \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \mathbf{u}(t) dt + \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \partial_t \mathbf{u}(s) ds$, we derive that

$$\begin{aligned}
& \sum_{n=0}^k \tau (\|\mathbf{u}^{n+1}\|_{L^4}^2 + \|\mathbf{u}^{n+2}\|_{L^4}^2) \leq 2\|\mathbf{u}\|_{L^2(0, t_{k+2}; L^4)}^2 + 2\tau^2 \|\partial_t \mathbf{u}\|_{L^2(0, t_{k+2}; L^4)}^2 \\
& \leq C\|\mathbf{u}\|_{L^2(0, t_{k+2}; H^1)}^2 + C\tau^2 \|\partial_t \mathbf{u}\|_{L^2(0, t_{k+2}; H^1)}^2 \\
(3.24) \quad & \leq C\Phi(M)^2.
\end{aligned}$$

We see that when

$$(3.25) \quad \tau < \frac{1}{8C_3\Phi(M)^8}$$

we have $\tau 2C_3\Phi(M)^6 (\|\mathbf{u}^{n+1}\|_{L^4}^2 + \|\mathbf{u}^{n+2}\|_{L^4}^2) < 1/2$, satisfying the condition of Lemma 3.5, and so

$$\begin{aligned}
& \max_{0 \leq n \leq k} \|\mathbf{e}_h^{n+1}\|_{L^2}^2 \leq \exp \left(C\Phi(M)^6 \sum_{n=0}^k \tau (\|\mathbf{u}^{n+1}\|_{L^4}^2 + \|\mathbf{u}^{n+2}\|_{L^4}^2) \right) C\Phi(M)^4(\tau^2 + h^2) \\
& \leq \exp(C\Phi(M)^8) C\Phi(M)^4(\tau^2 + h^2) \\
(3.26) \quad & \leq \exp(C_4\Phi(M)^8)(\tau^2 + h^2),
\end{aligned}$$

Substituting the estimate above into (3.23), we obtain

$$(3.27) \quad \max_{0 \leq n \leq k} \|\mathbf{e}_h^{n+1}\|_{L^2}^2 + \sum_{n=0}^k \tau \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 \leq \exp(C_5\Phi(M)^8)(\tau^2 + h^2).$$

From the last inequality we derive that, for any $0 \leq n \leq k$,

$$\|\mathbf{e}_{h,\tau}\|_{L^\infty(0, t_{n+1}; L^4)}^2 \leq \min(C_6 h^{-3/2} \|\mathbf{e}_h\|_{L^\infty(0, t_{n+1}; L^2)}^2, C_6 \tau^{-1} \|\nabla \mathbf{e}_h\|_{L^2(0, t_{n+1}; L^2)}^2)$$

$$\begin{aligned}
&\leq C_6 \min(h^{-3/2}, \tau^{-1}) \left(\max_{0 \leq n \leq k} \|\mathbf{e}_h^{n+1}\|_{L^2}^2 + \mu \sum_{n=0}^k \tau \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 \right) \\
(3.28) \quad &\leq C_6 \exp(C_5 \Phi(M)^8) (\tau + h^{1/2}),
\end{aligned}$$

and for any $t \in (t_n, t_{n+1}]$ we have

$$\begin{aligned}
&\|\mathbf{u}_h^n - \mathbf{u}(\cdot, t)\|_{L^2}^2 \\
&\leq 3\|\mathbf{e}_h^n\|_{L^2}^2 + 3\|R_h \mathbf{u}^n - \mathbf{u}^n\|_{L^2}^2 + 3\|\mathbf{u}^n - \mathbf{u}(\cdot, t)\|_{L^2}^2 \\
&\leq 3 \exp(C \Phi(M)^8) (\tau^2 + h^2) + C \|\mathbf{u}^n\|_{H^2}^2 h^4 + C \tau^2 \|\partial_t \mathbf{u}\|_{L^\infty(t_n, t_{n+1}; L^2)}^2 \\
(3.29) \quad &\leq \exp(C_7 \Phi(M)^8) (\tau^2 + h^2),
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathbf{u}_h^n - \mathbf{u}(\cdot, t)\|_{L^4}^2 \\
&\leq 3\|\mathbf{e}_h^n\|_{L^4}^2 + 3\|R_h \mathbf{u}^n - \mathbf{u}^n\|_{L^4}^2 + 3\|\mathbf{u}^n - \mathbf{u}(\cdot, t)\|_{L^4}^2 \\
&\leq C_6 \exp(C_5 \Phi(M)^8) (\tau + h^{1/2}) + C_8 \|\mathbf{u}^n\|_{H^2}^2 h^{5/2} + C_8 \tau \|\partial_t \mathbf{u}\|_{L^2(t_n, t_{n+1}; L^4)}^2 \\
(3.30) \quad &\leq \exp(C_9 \Phi(M)^8) (\tau + h^{1/2}),
\end{aligned}$$

which implies that

$$(3.31) \quad \|\mathbf{u}_{h,\tau} - \mathbf{u}\|_{L^\infty(0, t_{k+1}; L^2)}^2 \leq \exp(C_7 \Phi(M)^8) (\tau^2 + h^2),$$

$$(3.32) \quad \|\mathbf{u}_{h,\tau} - \mathbf{u}\|_{L^\infty(0, t_{k+1}; L^4)}^2 \leq \exp(C_9 \Phi(M)^8) (\tau + h^{1/2}).$$

When

$$(3.33) \quad \tau + h^{1/2} < \exp(-C_9 \Phi(M)^8),$$

we have

$$(3.34) \quad \|\mathbf{u}_{h,\tau} - \mathbf{u}\|_{L^\infty(0, t_{k+1}; L^4)} \leq 1,$$

and this completes the mathematical induction on (3.12).

Overall, by mathematical induction, when the mesh conditions (3.13), (3.25) and (3.33) are satisfied, we have

$$(3.35) \quad \|\mathbf{u}\|_{L^\infty(0, T; L^4)} \leq \|\mathbf{u}_{h,\tau}\|_{L^\infty(0, T; L^4)} + 1$$

and, as a consequence of Lemma 3.4, the solution \mathbf{u} possesses the regularity (2.10). Theorem 2.1 is proved with

$$(3.36) \quad \tau_M = \min \left(\frac{\varphi(M)}{2}, \frac{1}{8C_2 \Phi(M)^8}, \frac{1}{2} \exp(-C_9 \Phi(M)^8) \right)$$

and

$$(3.37) \quad h_M = \frac{1}{4} \exp(-2C_9 \Phi(M)^8).$$

Remark 3.2. Clearly, we can choose $C_9 \geq C_7$ in the analysis above. In this case, (3.31) and (3.33) imply the a posteriori error estimate:

$$(3.38) \quad \|\mathbf{u}_{h,\tau} - \mathbf{u}\|_{L^\infty(0, T; L^2)}^2 \leq \tau + h^{3/2}.$$

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